SOBOLEV SPACES AND SOBOLEV EMBEDDING THEOREM

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In this preliminary section we will not present carefully with all the details but briefly describe the related definitions and facts (from lectures) which will be used in the main sections of this paper. Let $\Omega \subset \mathbb{R}^n$ be open, we first recall some relevant spaces and norms:

1. Continuous Functions Spaces. For non negative integers m, let $C^m(\Omega)$ denote, as usual, the space of continuous functions with continuous derivative upto order m. $C^\infty = \cap_{m=1}^{\infty} C^m$. The subspaces C_0^m, C_0^∞ consists of functions in C^m, C^∞ with compact support. Define

$$C_B^m(\Omega) := \{\phi \in C^m(\Omega)|\ D^\alpha \phi \text{ is bounded for } |\alpha| \leq m\} \tag{0.1}$$

$C_B^m(\Omega)$ is a Banach space with norm given by [1].

$$||\phi; C_B^m(\Omega)|| = \sup_{0 \leq \alpha \leq m,\ x \in \Omega} |D^\alpha \phi(x)| \tag{0.2}$$

2. L^p spaces $1 \leq p < \infty$. Let $\Omega \subset \mathbb{R}^n$ open, let $1 \leq p < \infty$. We denote $L^p(\Omega)$ the class of all measurable functions u defined on Ω such that

$$||u||_p := \left(\int_\Omega |u(x)|^p dx \right)^{1/p} < \infty \tag{0.3}$$

We identify $L^p(\Omega)$ functions that are equal almost everywhere in Ω. Elements of L^p spaces are equivalent classes of functions such that (0.3) holds.

Remark 0.4. A function u that is defined almost everywhere in Ω is locally integrable ($u \in L^1_{loc}(\Omega)$) provided $u \in L^1(U)$ for every open $U \subsetneq \Omega$.

3. L^∞ space.

$$L^\infty(\Omega) := \{u \text{ measurable } : |u(x)| \leq K \text{ a.e. on } \Omega\} \tag{0.5}$$

We identify $L^\infty(\Omega)$ functions that are equal almost everywhere in Ω. The norm in L^∞ is given by

$$||u||_\infty := \inf\{K : |u(x)| \leq K \text{ a.e. on } \Omega\} \tag{0.6}$$

We here state without proof two (familiar) inequalities used in the discussion in this paper.

Theorem 0.7. ***(Hölder's Inequality)*** *Let $1 < p < \infty$ and $1/p + 1/p' = 1$. If $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$,then $uv \in L^1(\Omega)$ and*

$$||uv||_1 \leq ||u||_p ||v||_{p'} \tag{0.8}$$

Date: November 16, 2020.

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[1]The notation $||\cdot; X||$ serves to point out the norm of space X

The next Interpolation Inequality follows from Hölder's Inequality and is used in the proof of Sobolev Imbedding theorem.

Theorem 0.9. *Let $1 \leq p < q < r \leq \infty$ such that for some $0 < \theta < 1$,*

$$\frac{1}{q} = \frac{\theta}{q} + \frac{1-\theta}{r} \tag{0.10}$$

If $u \in L^p(\Omega) \cap L^r(\Omega)$ then $u \in L^q(\Omega)$ and

$$||u||_q \leq ||u||_p^\theta ||u||_r^{1-\theta} \tag{0.11}$$

Another consequence of Hölder's Inequality is the following lemma:

Lemma 0.12. *$L^p(\Omega) \subset L^1_{loc}(\Omega)$ for $1 \leq p \leq \infty$.*

Proof. The case when $p = 1$ holds trivially. For $u \in L^\infty(\Omega)$, any open $U \subsetneq \Omega$, we have:

$$\int_U |u(x)|dx \leq \left(\int_U 1dx\right) ||u||_\infty = C||u||_\infty \tag{0.13}$$

follows directly from the definition. For $1 < p < \infty$, $u \in L^p(\Omega)$, and $U \subsetneq \Omega$ open, Hölder inequality gives:

$$\int_U |u(x)|dx \leq \left(\int_U |u(x)|^p dc\right)^{1/p} \left(\int_U 1dx\right)^{1-1/p} \leq C||u(x)||_p \tag{0.14}$$

□

4. Distributions. We denote the space of test functions by $\mathscr{D}$ and the space of distributions by $\mathscr{D}'$. See [AF] for more details.

5. Imbedding. We say a normed space X is *imbedded* in the normed space Y, and we write $X \to Y$ to designate this imbedding, provided that

1. X is a vector subspace of Y.
2. The identity operator I defined on X into Y by $Ix = x$ for all $x \in X$ is continuous. We observe that continuity $\iff$

$$||Ix; Y|| \leq M||x; X||, \quad x \in X \tag{0.15}$$

since I is linear.

1. Sobolev Spaces

Definition 1.1. (Sobolev space) We define a functional $||\cdot||_{m,p}$ called *Sobolev norm*, where for any positive integer m and $1 \leq p \leq \infty$:

$$\begin{aligned} ||u||_{m,p} &= \left(\sum_{0\leq|\alpha|\leq m} ||D^\alpha u||_p^p\right)^{1/p} \quad \text{for } 1 \leq p < \infty \\ ||u||_{m,\infty} &= \sup_{0\leq|\alpha|\leq m} ||D^\alpha u||_\infty \end{aligned} \tag{1.2}$$

For any positive integer m and $1 \leq p \leq \infty$ we consider three vector spaces with norm $||\cdot||_{m,p}$:

1. $H^{m,p}(\Omega) :=$ the completion of $\{u \in C^m(\Omega) : ||u||_{m,p} < \infty\}$.

(2) $W^{m,p}(\Omega) := \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for } 0 \le |\alpha| \le m\}$, where $D^\alpha u$ is the weak partial derivative of u.
(3) $W_0^{m,p}(\Omega) :=$ the closure of $C_0^\infty(\Omega)$ in the space $W^{m,p}(\Omega)$.

These space with appropriate norm $||\cdot||_{m,p}$ are called *Sobolev spaces* over Ω.

Remark 1.3. By definition $W^{0,p}(\Omega) = L^p(\Omega)$. Moreover, for $1 \le p < \infty$, $W_0^{m,p}(\Omega) = L^p(\Omega)$ because C_0^∞ is dense in $L^p(\Omega)$.

Remark 1.4. We have imbeddings $W_0^{m,p}(\Omega) \to W^{m,p}(\Omega) \to L^p(\Omega)$ by definition.

Our main goal in this section is to show that the first two definitions of Sobolev space coincide, that is $H^{m,p} = W^{m,p}$.

Theorem 1.5. *$W^{m,p}(\Omega)$ is a Banach space.*

Proof. Let $\{u_n\}$ be a Cauchy sequence in $W^{m,p}(\Omega)$. We show that $\{u_n\}$ converges to some $u \in W^{m,p}(\Omega)$. We observe that by the definition of Sobolev norm,

$$||D^\alpha u_n||_p \le ||u_n||_{m,p} \tag{1.6}$$

for $0 \le |\alpha| \le m$, so $\{D^\alpha u_n\}$ is Cauchy in $W^{m,p}(\Omega)$ and thus Cauchy in $L^p(\Omega)$. Completeness of $L^p(\Omega) \Rightarrow$ there exists $u,\ u_\alpha$ such that $u_n \to u$, $D^\alpha u_n \to u_\alpha$ in $L^p(\Omega)$. Lemma 0.12 implies that $u, u_n, D^\alpha u_n, u_\alpha \in L^1_{loc}$. Now consider the distributions defined by $u,\ u_n$:

$$T_u(\phi) = \int_\Omega u(x)\phi(x)dx \quad T_{u_n}(\phi) = \int_\Omega u_n(x)\phi(x)dx \tag{1.7}$$

for $\phi \in \mathscr{D}(\Omega) := C_0^\infty(\Omega)$.

$$|T_{u_n}(\phi) - T_u(\phi)| \le \int_\Omega |u(x) - u_n(x)||\phi(x)|dx \le ||u - u_n||_p||\phi||_{p'} \tag{1.8}$$

by Hölder's inequality with $1/p + 1/p' = 1$. So $Tu_n \to Tu$ in $\mathscr{D}'(\Omega)$. Similarly $TD^\alpha u \to Tu_\alpha$ in $\mathscr{D}'(\Omega)$. So we have:

$$\begin{aligned} T_{u_\alpha}(\phi) &= \lim_{n\to\infty} T_{D^\alpha u}(\phi) = \lim_{n\to\infty} \int_\Omega D^\alpha u(x)\phi(x)dx = (-1)^{|\alpha|} \lim_{n\to\infty} \int_\Omega u(x)D^\alpha\phi(x)dx \\ &= (-1)^{|\alpha|} T_u(D^\alpha\phi) = T_{D^\alpha u}(\phi) \end{aligned} \tag{1.9}$$

where the the last two equality follow from integration by part. So $u_\alpha = D^\alpha u$ in the distributional sense for $|\alpha| \le m$. $D^\alpha u \in L^p(\Omega)$ implies $u \in W^{m,p}(\Omega)$. Finally, $D^\alpha u_n \to D^\alpha u$ implies that $||u_n - u||_{m,p} \to 0$. Thus, $u_n \to u$ and $W^{m,p}(\Omega)$ is complete. □

Corollary 1.10. *$H^{m,p}(\Omega) \subset W^{m,p}(\Omega)$*

Proof. From definition, every element of $S = \{u \in C^m(\Omega) : ||u||_{m,p} < \infty\}$ is in $W^{m,p}(\Omega)$. $W^{m,p}(\Omega)$ complete $\Rightarrow$ the completion of $S = H^{m,p}(\Omega) \subset W^{m,p}(\Omega)$. □

To prove the other direction of the containment that is $W^{m,p}(\Omega) \subset H^{m,p}(\Omega)$, we need to introduce the following tool:

Definition 1.11. (Mollifiers) Let J be a real-valued non negative function in $C_0^\infty(\mathbb{R}^n)$ such that:

(1) $J(x) = 0$ for $|x| \ge 1$ and
(2) $\int_{\mathbb{R}^n} J(x)dx = 1$.

Let $J_\epsilon(x) = \epsilon^n J(x/\epsilon)$ Then J_ϵ satisfies:

(1) $J_\epsilon(x) = 0$ for $|x| \geq \epsilon$ and
(2) $\int_{\mathbb{R}^n} J_\epsilon(x)dx = 1$.

Then $J_\epsilon(x)$ is called a *mollifier* and

$$J_\epsilon * u(x) := \int_{\mathbb{R}^n} J(x-y)u(y)dy \tag{1.12}$$

for any u that the integral is defined, is called *a mollification of regularization of u*.

Theorem 1.13. *If $u \in L^p(\Omega)$, $1 \leq p < \infty$ then $J_\epsilon * u \in L^p(\Omega)$, and*

$$a).||J_\epsilon * u||_p \leq ||u||_p, \qquad b). \lim_{\epsilon \to 0^+} ||J_\epsilon * u - u||_p = 0 \tag{1.14}$$

Proof. Let $u \in L^p(\Omega)$, for $1 < p < \infty$ and $1/p + 1/p' = 1$, we have:

$$\begin{aligned} \int_{\mathbb{R}^n} J(x-y)u(y)dy &= \int_{\mathbb{R}^n} J(x-y)^{1/p'} J(x-y)^{1/p} u(y)dy \\ &\leq \left(\int_{\mathbb{R}^n} J(x-y)dy\right)^{1/p'} \left(\int_{\mathbb{R}^n} J(x-y)|u(y)|^p dy\right)^{1/p} \\ &= \left(\int_{\mathbb{R}^n} J(x-y)|u(y)|^p dy\right)^{1/p} \end{aligned} \tag{1.15}$$

by Hölder inequality. So we have:

$$\begin{aligned} ||J_\epsilon * u||_p^p &= \int_{\mathbb{R}^n} |J_\epsilon * u(x)|^p dx \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J(x-y)|u(y)|^p dydx \\ &= \int_{\mathbb{R}^n} |u(y)|^p dy = ||u||_p^p \end{aligned} \tag{1.16}$$

using Fubini theorem and for any y, $\int_{\mathbb{R}^n} J(x-y)dx = 1$. We have proved (1.14) a). for $1 < p < \infty$. The case when $p = 1$ follows directly from the definition of convolution (1.12).

For (1.14) b), fix $\epsilon > 0$, since $C_0(\Omega)$ dense in L^p, we can choose $\phi \in C_0(\Omega)$ such that $||u - \phi||_p \leq \epsilon/3$. By (1.14) a), since $\phi \in L^1(\Omega)$, we also have $||J_\epsilon * u - J_\epsilon * \phi||_p \leq \epsilon/3$. Now consider

$$\begin{aligned} |J_\epsilon * \phi(x) - \phi(x)| &= |\int_{\mathbb{R}^n} J_\epsilon(x-y)\phi(y)dy - \phi(x)| \\ &= |\int_{\mathbb{R}^n} J_\epsilon(x-y)\phi(y)dy - \phi(x)\int_{\mathbb{R}^n} J_\epsilon(x-y)dy| \\ &= |\int_{\mathbb{R}^n} J_\epsilon(x-y)(\phi(y) - \phi(x))dy| \\ &\leq \sup_{|y-x|<\epsilon} |(\phi(y) - \phi(x)| \end{aligned} \tag{1.17}$$

We can choose a cover of the support of $\phi(x)$, $O_1, O_2, ..., O_n$ with Lesbegue measure $m(O_i) < 1/3^p$ (finite since compact) such that on each O_i, we have $\sup_{|y-x|<\epsilon} |(\phi(y) - \phi(x)| < \epsilon/2^i$

(uniform continuity of ϕ). Hence:

$$
\begin{aligned}
||J_\epsilon * \phi - \phi||_p &\le || \sup_{|y-x|<\epsilon} |\phi(y)-\phi(x)|||_p \\
&= \left(\int_{\mathbb{R}^n} \sup_{|y-x|<\epsilon} |\phi(y)-\phi(x)|^p dx \right)^{1/p} \\
&\le \left(\sum_{i=1}^{n} \int_{O_i} \sup_{|y-x|<\epsilon} |\phi(y)-\phi(x)|^p dx \right)^{1/p} \\
&\le \left(\sum_{i=1}^{n} \frac{1}{3^p} \frac{\epsilon^p}{2^{ip}} \right)^{1/p} \\
&< \frac{\epsilon}{3}
\end{aligned} \tag{1.18}
$$

We have by triangle inequality:

$$
||J_\epsilon * u - u||_p \le ||J_\epsilon * u - J_\epsilon * \phi||_p + ||J_\epsilon * \phi - \phi||_p + ||\phi - u||_p < \epsilon \tag{1.19}
$$

□

Lemma 1.20. *(Corollary) (**Mollification in** $W^{m,p}(\Omega)$) Let $u \in W^{m,p}(\Omega)$ for $1 \le p < \infty$. If $\Omega' \subset \Omega$ with compact closure in Ω, then*

$$
\lim_{\epsilon \to 0} J_\epsilon * u = u \text{ in } W^{m,p}(\Omega') \tag{1.21}
$$

Proof. Let $\epsilon < \text{dist}(\Omega', \partial\Omega)$, let $\tilde{u}$ be the zero extension of u outside of Ω. For $\phi \in \mathscr{D}(\Omega)$:

$$
\begin{aligned}
\int_{\Omega'} J_\epsilon * u(x) D^\alpha \phi(x) dx &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} J_\epsilon(y) \tilde{u}(x-y) D^\alpha \phi(x) dx dy \\
&= (-1)^{|\alpha|} \int_{\mathbb{R}^n} \int_{\Omega'} J_\epsilon(y) D_x^\alpha \tilde{u}(x-y) \phi(x) dx dy \\
&= (-1)^{|\alpha|} \int_{\mathbb{R}^n} J_\epsilon * D^\alpha u(x) \phi(x) dx
\end{aligned} \tag{1.22}
$$

Thus, $D^\alpha J_\epsilon * u(x) = J_\epsilon * D^\alpha u(x)$ in the distributional sense in Ω'. Since $D^\alpha u(x) \in L^p(\Omega)$ for $|\alpha| \le m$, we have by theorem 1.13 on Ω' :

$$
\lim_{\epsilon \to 0} ||D^\alpha J_\epsilon * u(x) - D^\alpha u(x)||_p = \lim_{\epsilon \to 0} ||J_\epsilon * D^\alpha u(x) - D^\alpha u(x)||_p = 0 \tag{1.23}
$$

This gives $||J_\epsilon * u(x) - u(x)||_{m,p} \to 0$. □

Theorem 1.24. *If $1 \le p < \infty$, then $H^{m,p} = W^{m,p}$*

Proof. By Corollary 1.10, it is left to show $W^{m,p}(\Omega) \subset H^{m,p}(\Omega)$. That is $\{\phi \in C^m(\Omega) : ||\phi||_{m,p} < \infty\}$ is dense in $W^{m,p}(\Omega)$. If $u \in W^{m,p}(\Omega)$, we in fact show $\exists \phi \in C^\infty(\Omega)$ such that $||u - \phi||_{m,p} < \epsilon$.

For $k = 1, 2, ...$, let

$$
\Omega_k := \{x \in \Omega : |x| < k, \ \text{dist}(x, \partial\Omega) > 1/k\} \tag{1.25}
$$

Define $\Omega_{-1} = \Omega_0 := \varnothing$. Then $\mathcal{O} = \{U_k : U_k = \Omega_{k+1} \cap (\overline{\Omega_{k-1}})^C\}$ covers Ω. Let Ψ be a partition of unity subordinate to this cover $\mathcal{O}$ and ψ_k denote the sum of the finite sum that has support in U_k. If $0 < \epsilon < \frac{1}{(k+1)(k+1)}$, $J_\epsilon * (\psi_k u)$ has support in $V_k := \Omega_{k+2} \cap \Omega_{k-2}{}^C$. By previous lemma, we can choose $\epsilon_k < \frac{1}{(k+1)(k+2)}$ such that:

$$||J_{\epsilon_k} * (\psi_k u) - (\psi_k u)||_{m,p,\Omega} = ||J_{\epsilon_k} * (\psi_k u) - (\psi_k u)||_{m,p,V_k} < \frac{\epsilon}{2^k} \tag{1.26}$$

On Ω_k, we have:

$$u(x) = \sum_{i=1}^{k+2} \psi_i u(x) \qquad \phi(x) := \sum_{i=1}^{k+2} J_{\epsilon_i} * \psi_i u(x) \tag{1.27}$$

We observe that both of the above sum is finite for $x \in \Omega_k$ for each k by the definition of ψ_i and J_{ϵ_i}. Also, $\phi(x) \in C^\infty(\Omega)$ since each term of the sum is. This gives:

$$||u(x) - \phi(x)||_{m,p,\Omega_k} \leq \sum_{i=1}^{k+2} ||J_{\epsilon_i} * \psi_i u(x) - \psi_i u(x)||_{m,p,\Omega} < \epsilon \tag{1.28}$$

By the monotone convergence theorem as $k \to \infty$ we have:

$$||u(x) - \phi(x)||_{m,p,\Omega} \leq \sum_{i=1}^{\infty} ||J_{\epsilon_i} * \psi_i u(x) - \psi_i u(x)||_{m,p,\Omega} < \epsilon \tag{1.29}$$

□

Remark 1.30. We give an example that above theorem fails when $p = \infty$. Let $\Omega = (-1, 1) \subset \mathbb{R}$, and $u(x) = |x|$. Then $u'(x) = x/|x|$ for $x \neq 0$ and so $u \in W^{1,\infty}(\Omega)$. But $u \neq H^{1,\infty}$. In fact, there is no $\phi \in C^1(\Omega)$ such that $||\phi'(x) - u'(x)||_\infty < 1/2$.

2. Sobolev Imbeddings

In this section, we explore the imbedding properties of Sobolev spaces. The imbedding properties of Sobolev spaces depend on the regularity properties of Ω which is normally expressed in terms of geometric or analytic conditions. We first introduce some relevant definitions:

Definition 2.1. (Finite Cone) Let $v \in \mathbb{R}^n$ be a nonzero vector. For each $x \neq 0$, let $\angle(x, v)$ be the angle between x, v. For given $v, \rho > 0$, and $0 < \kappa \leq \pi$, the set

$$C = \{x \in \mathbb{R}^n : x = 0 \text{ or } 0 < |x| \leq \rho, \angle(x, v) \leq \kappa/2\} \tag{2.2}$$

is called a *finite cone* of height ρ, axis direction v and aperture angle κ with vertex at the origin. We can define a finite cone with arbitrary vertex by a translation of C: i.e. $x_0 + C := \{x_0 + y : y \in C\}$ is a finite cone at vertex x_0.

Definition 2.3. (Cone Condition) Ω satisfies the *cone condition* if there exists a finite cone C such that each $x \in \Omega$ is the vertex of a finite cone C_x contained in Ω and congruent to C. This means that C_x can be obtained by a rigid motion (translation, rotation, reflection) of C.

There are several versions of Sobolev Imbedding Theorem with different target spaces of the imbedding. In this section, we only consider the most elementary case, stated as the following theorem:

Theorem 2.4. ***Sobolev Imbedding Theorem*** *Let Ω be a domain in $\mathbb{R}^n$, for $1 \leq k \leq n$, let Ω_k be the intersection of Ω with a plane of dimension k in $\mathbb{R}^n$. Let $j \geq 0$, $m \geq 1$ be integers, $1 \leq p < \infty$. Suppose Ω satisfies the cone condition, then:*

(1) If either $mp > n$ or $m = 1$, $p = 1$, then

$$W^{j+m,p}(\Omega) \to C^j_B(\Omega) \tag{2.5}$$

where we recall $C^j_B(\Omega)$ as in (0.1).

(2) If moreover $1 \leq k \leq n$, then

$$W^{j+m,p}(\Omega) \to W^{j,q}(\Omega_k) \qquad p \leq q \leq \infty \tag{2.6}$$

in particular,the case when $j = 0$, $k = n$ is

$$W^{m,p}(\Omega) \to L^q(\Omega) \qquad p \leq q \leq \infty \tag{2.7}$$

Since elements of $W^{m,p}$ are not function defined everywhere, but equivalence classes of such functions defined almost everywhere, we need to explain what is meant by the imbedding of $W^{m,p}$ into continuous function space. What this means is that an element $u \in W^{m,p}$ should contain an element $\tilde{u}(x)$ in it's "equivalent class" ($u = \tilde{u}$ a.e.) that is in the continuous function space, the target of the imbedding, and is bounded by $K||\tilde{u}(x)||_{m,p}$ for some constant K.

To make sense of the imbedding $W^{j+m,p}(\Omega) \to W^{j,q}(\Omega_k)$ where $k < n$, we observe that by theorem 1.24, $u \in W^{j+m,p}(\Omega)$ is the limit of a sequence of continuous functions $\{u_i\} \in C^m$. Each of u_i restrict to Ω_k (called traces) belongs to $C^m(\Omega_k)$. The imbedding implies that these traces converge in $W^{j,q}(\Omega_k)$ to a function $\tilde{u}$ that is independent of the choice of $\{u_i\}$, and $||\tilde{u}||_{j,q,\Omega_k} \leq K||u||_{j+m,p,\Omega}$.

The main tool in proving the Sobolev imbedding theorem is the following local estimate.

Lemma 2.8. *Let domain $\Omega \subset \mathbb{R}^n$ satisfy the cone condition. There exists a constant K depending on m, n and the dimensions ρ, κ of the cone C, such that for every $u \in C^\infty(\Omega)$, every $x \in \Omega$, and every r such that $0 < r \leq \rho$, we have:*

$$|u(x)| \leq K \left(\sum_{|\alpha| \leq m-1} r^{|\alpha|-n} \int_{C_{x,r}} |D^\alpha u(y)| dy + \sum_{|\alpha|=m} \int_{C_{x,r}} |D^\alpha u(y)||x-y|^{m-n} dy \right) \tag{2.9}$$

where $C_{x,r} = \{y \in C_x : |x-y| \leq r\}$ and $C_x \subset \Omega$ is the cone congruent to C with vertex at x.

Proof. Apply to Taylor's theorem with remainder to the function $f(t) = u(tx + (1-t)y)$, $x \in \Omega$, $y \in C_{x,r}$.

$$f(1) = \sum_{j=0}^{m-1} \frac{1}{j!} f^{(j)}(0) + \frac{1}{(m-1)!} \int_0^1 (1-t)^{m-1} f^{(m)}(t) dt \tag{2.10}$$

the derivative of f given by:

$$f^{(j)}(t) = \sum_{|\alpha|=j} \frac{j!}{\alpha!} D^\alpha u(tx + (1-t)y)(x-y)^\alpha \tag{2.11}$$

where α is the usual multi index: $\alpha! = \alpha_1!...\alpha_n!$, $(x-y)^\alpha = (x_1 - y_1)^\alpha_1 ...(x_n - y_n)^\alpha_n$.

We observe that $f(1) = x, f(0) = y$, so plugging (2.11) to (2.10) gives:

$$|u(x)| \le \sum_{|\alpha|\le m-1} \frac{1}{\alpha!}|D^\alpha u(y)||x-y|^{|\alpha|} + \sum_{|\alpha|=m} \frac{m}{\alpha!}|x-y|^m \int_0^1 (1-t)^{m-1}|D^\alpha u(tx+(1-t)y)|dt \tag{2.12}$$

We integrate y over $C_{x,r}$ on both sides. Using that $C_{x,r}$ has volume cr^n for a constant c, and $|x-y| \le r$ for $y \in C_{x,r}$, we have:

$$\begin{aligned} cr^n|u(x)| &\le \sum_{|\alpha|\le m-1} \frac{r^{|\alpha|}}{\alpha!} \int_{C_{x,r}} |D^\alpha u(y)|dy \\ &+ \sum_{|\alpha|=m} \frac{m}{\alpha!} \int_{C_{x,r}} |x-y|^m \int_0^1 (1-t)^{m-1}|D^\alpha u(tx+(1-t)y)|dtdy \end{aligned} \tag{2.13}$$

Now the first integral is in the form desired. For the second double integral, let $z = tx + (1-t)y$, using $z - x = (1-t)(y-x)$, $dz = (1-t)^n dy$, and changing the order of integration we obtain:

$$\int_0^1 (1-t)^{-n-1} \int_{C_{x,(1-t)r}} |z-x|^m|D^\alpha u(z)|dzdt \tag{2.14}$$

Change in order of integration again we have:

$$\begin{aligned} &\int_{C_{x,r}} |z-x|^m|D^\alpha u(z)| \int_0^{1-(|z-x|/r)} (1-t^{-n-1})dtdz \\ &\le \frac{r^n}{n} \int_{C_{x,r}} |z-x|^{m-n}|D^\alpha u(z)|dz \end{aligned} \tag{2.15}$$

Now relable z as y above, together with (2.13) yields the inequality in the lemma. □

Now we're ready to prove the Sobolev Imbedding Theorem:

Proof. Theorem 2.4.

Step 0. We observe that it is sufficient to prove the case when $j = 0$. The general case follows by applying the special case to $D^\alpha u$.

For example , if $W^{m,p}(\Omega) \to L^q(\Omega)$ is proven, for $u \in W^{j+m,p}$, $D^\alpha u \in W^{m,p}$ for $|\alpha| \le j$, we have:

$$||u||_{j,q} = \left(\sum_{|\alpha|\le j} ||D^\alpha u||_{0,q}^q\right)^{1/q} \le K \left(\sum_{|\alpha|\le j} ||D^\alpha u||_{m,p}^p\right)^{1/p} \le K_1||u||_{j+m,p} \tag{2.16}$$

Step 1. Let $u \in W^{m,p}(\Omega) \cap C^\infty(\Omega)$, for $x \in \Omega$, we show

$$|u(x)| \le K||u||_{m,p} \tag{2.17}$$

For $p = 1, m = n$, (2.9) gives (2.17). If $p > 1, mp > n$, (2.9) $r = \rho$ gives:

$$|u(x)| \le K \left(\sum_{|\alpha|\le m-1} \rho^{|\alpha|-n} \int_{C_{x,\rho}} |D^\alpha u(y)|dy + \sum_{|\alpha|=m} \int_{C_{x,\rho}} |D^\alpha u(y)||x-y|^{m-n}dy\right) \tag{2.18}$$

By Hö*lder*'s inequality the first integral satisfies:
(2.19)

$$\int_{C_{x,\rho}} |D^\alpha u(y)|dy \leq ||D^\alpha u||_{p,C_{x,\rho}}||1||_{p',C_{x,\rho}} = ||D^\alpha u||_{p,C_{x,\rho}} c^{1/p'}\rho^{n/p'} \leq C||D^\alpha u||_{p,C_{x,\rho}}$$

where c denote the volume of $C_{x,1}$. Recall Vol$C_{x,\rho} = c\rho^n$. Using By Hö*lder* on the second integral in (2.18) gives:
(2.20)

$$\int_{C_{x,\rho}} |D^\alpha u(y)||x-y|^{m-n}dy \leq ||D^\alpha u||_{p,C_{x,\rho}} \left(\int_{C_{x,\rho}} |x-y|^{(m-n)p'}dy \right)^{1/p'} \leq C||D^\alpha u||_{p,C_{x,\rho}}$$

because the integral on the right is finite since $(m-n)p' > -n$ when $mp > n$. Using (2.19) and (2.20) in (2.18), we have:

$$|u(x)| \leq K \sum_{|\alpha|\leq m} ||D^\alpha u||_{p,C_{x,\rho}} \leq K||u||_{m,p} \tag{2.21}$$

because $C_{x,\rho} \subset \Omega$.

Step 2. To finish the proof of the first statement of the theorem, it's left to see that every element (the equivalent class) $u \in W^{m,p}(\Omega)$ has an element in $\tilde{u} \in C_B^0(\Omega)$. By theorem 1.24, every element of $u \in W^{m,p}(\Omega)$ is a limit of a Cauchy sequence of continuous functions. (2.17) implies that this sequence converge to a continuous function in $W^{m,p}(\Omega)$, so the limit has to equal to u almost everywhere on Ω. Thus, $u \in C_B^0(\Omega)$.

Step 3. Let Ω_k denote the intersection of Ω and a k-dimensional plane H. Let $\Omega_{k,\rho} := \{x \in \Omega : \text{dist}(x, \Omega_k) < \rho\}$. We extend u and its derivatives to be 0 outside Ω. We observe that the cone $C_{x,\rho} \subset B(x,\rho)$, the ball centered at x with radius ρ. Now integrating the p-th power of (2.21) gives:

$$\begin{aligned} \int_{\Omega_k} |u(x)|^p dx' &\leq K \sum_{|\alpha|\leq m} \int_{B(x,\rho)} \int_{\Omega_k} |D^\alpha u(y)|^p dy dx' \\ &= K \sum_{|\alpha|\leq m} \int_{\Omega_{k,\rho}} \int_{H\cap B(y,\rho)} |D^\alpha u(y)|^p dx' dy \\ &\leq K_1 ||u||^p_{m,p,\Omega} \end{aligned} \tag{2.22}$$

where dx' denotes the measure on the subspace Ω_k. (2.22) shows that $W^{m,p}(\Omega) \to W^{0,p}(\Omega_k) = L^p(\Omega_k)$ (recall the discussion after theorem 2.4). (2.17) gives $W^{m,p}(\Omega) \to W^{0,\infty}(\Omega_k) = L^\infty(\Omega_k)$. Finally Theorem 0.9 gives the imbedding $W^{m,p}(\Omega) \to W^{0,q}(\Omega_k) = L^q(\Omega_k)$ for $p \leq q \leq \infty$, which finishes the proof of the theorem. □

3. APPENDIX

The details filled by me :

Remark 0.4, lemma 0.12 and its proof.

The proof of Cor 1.10.

The details of proof of theorem 1.13 is added by me, including the correction (typo in Adams $du \to dy$) of (1.17), adding the intermediate steps of the string of equations, and adding the discussion below (1.17) for showing $||J_\epsilon * \phi - \phi||_p < \epsilon/3$.

In general, I provided more explanations and intermediate steps in most step of proofs than presented by Adams.

4. Bibliography

[AF] Adams, R. and Fournier, J., 2006. Sobolev Spaces. Oxford: Elsevier.