THE HILLE-YOSIDA THEOREM

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1. Semigroup of Operators

1.1. **Semigroup Operators.**

Definition 1.1. Let X be a Banach space. $Q(t)$, $t \in [0, +\infty)$ is a family of bounded linear operators over X that satisfies:

(1) $Q(0) = I$
(2) $Q(s+t) = Q(s)Q(t)$ for $s, t \geq 0$.

We say $Q(t)$ is a **(one-parameter) semigroup** of operators. We say $Q(t)$ is **strongly continuous** if it also satisfies:

(3) $\lim_{t\to 0} ||Q(t)x - x|| = 0$ for every $x \in X$.

We can associate with $\{Q(t)\}$ the operator A_ϵ by

$$A_\epsilon x = \frac{1}{\epsilon}[Q(\epsilon)x - x] \quad x \in X, \epsilon > 0 \tag{1.2}$$

Definition 1.3. The **infinitesimal generator** A is defined by

$$Ax = \lim_{\epsilon \to 0} A_\epsilon x \tag{1.4}$$

The **domain** $\mathcal{D}(A)$ is the set of all x where the above limit exists.

Clearly $\mathcal{D}(A)$ is a subspace of X. We check that A is a linear operator. For $x_1, x_2 \in \mathcal{D}(A)$, by strong continuity as $\epsilon \to 0$, $Q(\epsilon)x_1 + Q(\epsilon)x_2 \to x_1 + x_2$, and $Q(\epsilon)(x_1 + x_2) \to x_1 + x_2$, so $\lim_{\epsilon\to 0} A_\epsilon(x_1 + x_2) = \lim_{\epsilon\to 0} A_\epsilon(x_1) + A_\epsilon(x_2)$, that is $A(x_1 + x_2) = A(x_1) + A(x_2)$. $A(\alpha x) = \alpha A(x)$ for $\alpha \in \mathbb{C}$ follows from the same argument.

Given the definition of the infinitesimal generator, it is natural to ask when an operator is the infinitesimal generator of such a semigroup. This is answered in theorem 1.39.

Before stating the properties of $\{Q(t)\}$, we recall Banach Steinhaus Theorem from class without proof, which will be used in the proof of the following big theorem.

Theorem 1.5. *(Banach Steinhaus) Let V, W be Banach spaces. Let $T_j \in L(V, W)$ for $j = 1, 2, 3, ...$ Assume that for each $v \in V$, $\{T_j v\}$ is bounded for all j. (i.e. $\exists C_v$ such that $|T_j v| \leq C_v$.) Then $\{||T_j||\}$ is bounded for all j. (i.e. $\exists C$ such that $||T_j|| \leq C\ \forall j$.)*

The definition of equicontinuity and the following two lemma will also be used in proving theorem 1.10 and 1.39:

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Definition 1.6. Let X and Y be topological vector spaces and Γ a collection of linear maps from X to Y. We say Γ is **equicontinuous** if for every neighborhood W of 0 in Y there corresponds a neighborhood V of 0 in X such that $\Gamma(V) \subset W$ for all $\Lambda \in \Gamma$.

Lemma 1.7. *Let X, Y be topological vector spaces, $E \in \mathcal{B}(X,Y)$ is equicontinuous $\Leftrightarrow \exists M < \infty$ such that $||\Lambda|| \leq M \quad \forall \Lambda \in E$.*

Lemma 1.8. *Let X, Y be topological vector spaces, Y is a Frechet space. $\{\Lambda_n\}$ be an equicontinuous sequence of linear mappings from X to Y. If $\{\Lambda_n\}$ converges on some dense subset of X, it converges on all X, and the limit is continuous, that is*

$$\Lambda(x) = \lim_{n\to\infty} \Lambda_n(x) \tag{1.9}$$

and $\Lambda(x)$ is continuous.

Now we are ready to state the first main theorem of strongly continuous semigroup operators.

Theorem 1.10. *If the semigroup $\{Q(t)\}$ is strongly continuous, then:*

1. *There are constants C, γ such that*
$$||Q(t)|| \leq Ce^{\gamma t} \quad 0 \leq t \leq \infty \tag{1.11}$$
2. *$t \to Q(t)x$ is a continuous map of $[0,\infty)$ into X, for every $x \in X$.*
3. *$\mathcal{D}(A)$ is dense in X and A is closed.*
4. *For every $x \in \mathcal{D}(A)$, we have*
$$\frac{d}{dt}Q(t)x = AQ(t)x = Q(t)Ax \tag{1.12}$$
5. *For every $x \in X$,*
$$Q(t)x = \lim_{\epsilon\to 0}(\exp(tA_\epsilon))x \tag{1.13}$$
where the convergence is uniform on every compact subset of $[0,\infty)$.
6. *If $\lambda \in \mathbb{C}$ and $Re\lambda > \gamma$. the integral:*
$$R(\lambda)x = \int_0^\infty e^{-\gamma t}Q(t)xdt \tag{1.14}$$
*defines an operator $R(\lambda) \in \mathcal{B}(X)$(bounded operators $X \to X$), called the **resolvent** of $\{Q(t)\}$, whose range is $\mathcal{D}(A)$ and which inverts $\lambda I - A$.*

Proof. (1) Suppose there exists a sequence $(t_n) \to 0$ such that $||Q(t_n)|| \to \infty$. Banach Steinhaus theorem (using the contrapositive) implies that $\exists x \in X$ such that $\{||Q(t_n)x||\}$ is unbounded, which is a contradiction to to the assumption that $Q(t)$ is strong continuous (i.e. $\lim_{t\to 0} ||Q(t)x - x|| = 0$). Hence, there exists C and $\delta > 0$ such that $||Q(t)|| < C$ on $[0,\delta]$. Now, if $t \in [0,+\infty)$, we pick $n \in \mathbb{N}$ such that $(n-1)\delta \leq t < n\delta$, then $||Q(t/n)|| < C$.

$$||Q(t)|| = ||Q(n \cdot \frac{t}{n})|| = ||(Q(\frac{t}{n}))^n|| \leq ||Q(t)||^n \leq C^n \leq C^{1+t/\delta} \tag{1.15}$$

Finally, choose $\gamma = \log C^{1/\delta}$, we have $||Q(t)|| \leq Ce^{\gamma t}$

(2) Let $0 \leq s < t \leq T$, then

$$\begin{aligned} ||Q(t)x - Q(s)x|| &= ||Q(s+t-s)x - Q(s)x|| = ||Q(s)(Q(t-s)x - Ix)|| \\ &\leq ||Q(s)||\ ||(Q(t-s)x - Ix)|| \\ &\leq Ce^{\gamma T}||(Q(t-s)x - Ix)|| \end{aligned} \tag{1.16}$$

We note that the right hand side tends to zero when $t-s \to 0$ since $\lim_{t\to 0} ||Q(t)x - x|| = 0$, which proves the continuity.

(3) Since the previous part, we can define the integral

$$M_t x := \frac{1}{t}\int_0^t Q(s)x\ ds\ (x \in X,\ t > 0) \tag{1.17}$$

We note that $M_t \in \mathcal{B}(X)$ and $||M_t|| \leq Ce^{\gamma t}$ by part (1) of this theorem. We claim:

$$A_\epsilon M_t x = A_t M_\epsilon x \quad (\epsilon, t > 0,\ x \in X) \tag{1.18}$$

To prove the claim, we consider the equation:

$$\int_\epsilon^{\epsilon+t} Q(s)x - \int_0^t Q(s)x = \int_t^{\epsilon+t} Q(s)x - \int_0^\epsilon Q(s)x \tag{1.19}$$

By a change of variable, we have the left hand side of 1.19 equals

$$\begin{aligned} \int_0^t [Q(\epsilon+s) - Q(s)]x &= \int_0^t [Q(\epsilon)Q(s) - Q(s)]x \\ &= ((Q(\epsilon) - I)t)(\frac{1}{t}\int_0^t Q(s)x) = \epsilon A_\epsilon t M_t x \end{aligned} \tag{1.20}$$

The right hand side of 1.19 equals to:

$$\begin{aligned} \int_0^\epsilon [Q(t+s) - Q(s)]x &= \int_0^\epsilon [Q(t)Q(s) - Q(s)]x \\ &= ((Q(t) - I)\epsilon)(\frac{1}{\epsilon}\int_0^\epsilon Q(s)x) = tA_t\epsilon M_\epsilon x \end{aligned} \tag{1.21}$$

Above calculations proves the claim 1.18. We also have:

$$||M_t x - x|| = ||\frac{1}{t}\int_0^t (Q(s) - I)x\ ds|| \leq ||\sup_{[0,t]} Q(s)x - Ix|| \to 0 \tag{1.22}$$

Thus, as $\epsilon \to 0$, $A_t M_\epsilon x \to A_t x$, so $A_\epsilon M_t x \to A_t$. This shows that $M_t x \in \mathcal{D}(A)$. Since $M_\epsilon x \to x$, $\mathcal{D}(A)$ is dense in X. Moreover, we have

$$AM_t x = \lim_{\epsilon\to 0} A_\epsilon M_t x = \lim_{\epsilon\to 0} A_t M_\epsilon x = A_t x \tag{1.23}$$

To show A is a close map, suppose $x_n \in \mathcal{D}(A)$, $x_n \to x$, and $Ax_n \to y$. Since $Q(s), Q(t)$ commute, A_ϵ and M_t commute, and therefore A commutes with M_t on $\mathcal{D}(A)$. 1.23 gives:

$$A_t x_n = AM_t x_n = M_t A x_n \tag{1.24}$$

As $n \to \infty$, we have on the one hand $A_t x_n \to A_t x$, and on the other hand $M_t A x_n \to M_t y$. So $A_t x = M_t y$. As $t \to 0$. $M_t y \to y$. This shows that the limit of the left hand side exists i.e. $x \in \mathcal{D}(A)$ and $Ax = y$, which finishes this part of the proof.

(4) Multiply t on both sides of 1.23 Gives

$$A\int_0^t Q(s)x\ ds = Q(t)x - x \tag{1.25}$$

Since $Q(s)$ is continuous, we can differentiate both sides with respect to t, which gives us *(4)*. ($Q(t)Ax = AQ(t)x$ since $Q(t)A_\epsilon = A_\epsilon Q(t)$)

(5) We first need an estimate of $\exp\{tA_\epsilon\}$:

$$\begin{aligned} ||\exp\{tA_\epsilon\}|| &= ||e^{-t/\epsilon}\exp\{\frac{t}{\epsilon}Q(\epsilon)\}|| = ||e^{-t/\epsilon}\sum_{n=0}^{\infty}\frac{t^nQ(n\epsilon)}{\epsilon^n n!}|| \\ &\le e^{-t/\epsilon}\sum_{n=0}^{\infty}\frac{t^n||Q(n\epsilon)||}{\epsilon^n n!} \le e^{-t/\epsilon}\sum_{n=0}^{\infty}\frac{t^nCe^{\gamma\epsilon n}}{\epsilon^n n!}\ \textit{(by part (1))} \\ &= Ce^{-t/\epsilon}\exp(\frac{te^{\gamma\epsilon}}{\epsilon}) = C\exp(\frac{t}{\epsilon}(e^{\gamma\epsilon}-1)) \end{aligned} \tag{1.26}$$

For $0 < \epsilon \le 1$, we claim $C\exp(\frac{t}{\epsilon}(e^{\gamma\epsilon}-1)) < C\exp(te^{\gamma})$. To see this, we show

$$\frac{t}{\epsilon}(e^{\gamma\epsilon}-1) < te^{\gamma} \Leftrightarrow te^{\gamma\epsilon} - t < \epsilon te^{\gamma} \tag{1.27}$$

We denote the LHS, RHS by $f(\epsilon)$, $g(\epsilon)$ resp. First we notice that $f(0) = g(0)$. Taking ϵ derivative on both sides gives $f'(\epsilon) = t\gamma e^{\gamma\epsilon}$ and $g'(\epsilon) = te^{\gamma}$. Now notice again that $f'(1) = g'(1)$. We check that $f''(\epsilon) = t\gamma^2e^{\gamma\epsilon} > 0$, $g''(\epsilon) = 0 \Rightarrow f'(\epsilon) < g'(\epsilon) \Rightarrow f(\epsilon) < g(\epsilon)$ and the claim is proved. Above discussion gives for $0 < \epsilon < 1$:

$$||\exp\{tA_\epsilon\}|| \le C\exp(te^{\gamma}) \tag{1.28}$$

Now for fixed $x \in X$, we define:

$$\phi(s) = (\exp((t-s)A_\epsilon))Q(s)x \quad (0 \le s \le t) \tag{1.29}$$

If $x \in \mathcal{D}(A)$, *part (4)* of this theorem gives:

$$\phi'(s) = (\exp((t-s)A_\epsilon)Q(s)(Ax - A_\epsilon x) \tag{1.30}$$

$$\begin{aligned} ||\phi'(s)|| &\le ||(\exp((t)A_\epsilon)||\ ||Q(t)||\ ||(Ax - A_\epsilon x)|| \\ &\le C\exp(te^{\gamma})Ce^{\gamma t}||(Ax - A_\epsilon x)|| \equiv K(t)||(Ax - A_\epsilon x)|| \end{aligned} \tag{1.31}$$

where $K(t)$ is a constant that depend on t, $0 < \epsilon \le 1$. We note that $\phi(t) = Q(t)x$, and $\phi(0) = \exp(tA_\epsilon)x$. Fundamental theorem of calculus implies:

$$||Q(t)x - \exp(tA_\epsilon)x|| = ||\phi(t) - \phi(0)|| = ||\int_0^t \phi'(s)ds|| \le tK(t)||(Ax - A_\epsilon x)|| \tag{1.32}$$

If $x \in \mathcal{D}(A)$, taking $\epsilon \to 0$ proves the statement in the theorem.
To prove for all $x \in X$, we first note that $||Q(t) - \exp(tA_\epsilon)||$ is bounded for $0 < t \le T, 0 < \epsilon \le 1$ since $||Q(t)||$ and $||\exp(tA_\epsilon)||$ are. So these operators form an equicontinuous family of operators by 1.7. It follows that their convergence on the dense set $(D)(A)$ forces their convergence of all $x \in X$ by 1.8, which finishes the proof of this part of the theorem.

(6) We first have $||R(\gamma)|| \leq C \int_0^\infty e^{(\gamma - Re\lambda)}tdt = C\frac{1}{Re\lambda - \gamma} < \infty$. So $R(\gamma)$ is bounded. We calculate $\epsilon A_\epsilon R(\gamma)x$:

$$\begin{aligned} \epsilon A_\epsilon R(\gamma)x &= \int_0^\infty e^{-\gamma t}(Q(\epsilon) - I)Q(t)xdt = \int_0^\infty e^{-\gamma t}(Q(\epsilon + t) - Q(t))xdt \\ &= \int_0^\infty e^{-\gamma t}Q(\epsilon + t)xdt - \int_0^\infty e^{-\gamma t}Q(t)xdt \end{aligned} \tag{1.33}$$

Replace t with $t - \epsilon$ to the first integral, and applying integration by parts, we have:

$$\begin{aligned} A_\epsilon R(\gamma)x &= \frac{1}{\epsilon}e^{-\epsilon\gamma}\int_\epsilon^\infty e^{-\gamma t}Q(t)x\ dt - \frac{1}{\epsilon}\int_0^\infty e^{-\gamma t}Q(t)xdt \\ &= \frac{1}{\epsilon}e^{-\epsilon\gamma}\left(\int_0^\infty e^{-\gamma t}Q(t)x\ dt - \int_0^\epsilon e^{-\gamma t}Q(t)x\ dt\right) - \frac{1}{\epsilon}\int_0^\infty e^{-\gamma t}Q(t)xdt \\ &= \frac{1}{\epsilon}(e^{\epsilon\gamma} - 1)R(\gamma)x - \frac{1}{\epsilon}e^{\epsilon\gamma}\int_0^\epsilon e^{-\gamma t}Q(t)x\ dt \end{aligned} \tag{1.34}$$

As $\epsilon \to 0$ 1.22 shows that the second integral $\to x$. Below calculations show that $\frac{1}{\epsilon}(e^{\epsilon\gamma} - 1)R(\gamma)x \to \gamma$:

$$\frac{1}{\epsilon}(e^{\epsilon\gamma} - 1) = \frac{1}{\epsilon}(1 + \epsilon\lambda + o(\lambda\epsilon) - 1) \to \lambda \tag{1.35}$$

Thus, the right hand side of 1.34 converges to $\lambda R(\lambda)x - x$. So $R(\gamma)x \in \mathcal{D}(A)$. Moreover, we notice that $A_\epsilon R(\gamma)x \to AR(\gamma)x$ by definition. So we have

$$(\lambda I - A)R(\gamma)x = x \tag{1.36}$$

On the other hand, if $x \in \mathcal{D}(A)$, we have:

$$R(\lambda)A_\epsilon x = \int_0^\infty e^{-\gamma t}Q(t)A_\epsilon x\ dt \tag{1.37}$$

Taking the limit as $\epsilon \to 0$, use $Q(t)Ax = \frac{d}{dt}Q(t)x$ and integration by part, we have:

$$\begin{aligned} R(\lambda)Ax &= \int_0^\infty e^{-\gamma t}Q(t)Ax\ dt \\ &= e^{-\gamma t}Q(t)\ |_0^\infty - \int_0^\infty (-\gamma)e^{-\gamma t}Q(t)x\ dt \\ &= -x + \lambda R(\lambda)x \end{aligned} \tag{1.38}$$

This gives us $R(\lambda)(\lambda I - A)x = x$. Moreover, $\mathcal{D}(A)$ lies in the range of $R(\lambda)$ completing the proof.

□

Now the next theorem gives conditions when an operator is the infinitesimal generator of a semigroup.

Theorem 1.39. ***(Hille-Yosida)*** *A densely defined operator A in a Banach space X is the inifinitesimal generator of a strongly continuous semigroup $\{Q(t)\}$ $\Leftrightarrow$ there are constants C, γ such that*

$$||(\lambda I - A)^{-m}|| \leq C(\lambda - \gamma)^{-m} \tag{1.40}$$

for all $\lambda > \gamma$ and all $m \in \mathbb{N}$.

Proof. By part (6) of the previous theorem, we have $(\lambda I - A)^{-1} = R(\lambda)$ for $\lambda > \gamma$ where $R(\lambda)x = \int_0^\infty e^{-\gamma t} Q(t)x\ dt$ which is the Laplace transform of $Q(t)x$. Thus, $R^2(\lambda)x$ is the transform of the convolution: $R^2(\lambda)x = \int_0^\infty Q(t-s)Q(s)x\ ds = tQ(t)x$. Continuing this way, we have:

$$R(\lambda)^m x = \frac{1}{(m-1)!}\int_0^\infty t^{m-1} e^{-\lambda t} Q(t) x dt \tag{1.41}$$

for $m = 1,\ 2,\ 3,\ ...$ Therefore, we have estimates:

$$||R(\lambda)^m|| \le ||\frac{C}{(m-1)!}\int_0^\infty t^{m-1} e^{-\lambda t} e^{-\gamma t} dt|| = C(\lambda - \gamma)^{-m} \tag{1.42}$$

This shows the $\Rightarrow$ direction of the theorem.

Next, we set $S(\epsilon) = (I - \epsilon A)^{-1}$. For $0 < \epsilon < \epsilon_0 = 1/\lambda$. Then by assumption $||(\lambda I - A)|| \le C(\lambda - \gamma)^{-m}$, we have:

$$||S(\epsilon)|| \le C(1 - \epsilon\gamma)^{-m} \tag{1.43}$$

for $m = 1,\ 2,\ 3,\ ....$ We also have by definition:

$$S(\epsilon)(I - \epsilon A)x = x = (I - \epsilon A)S(\epsilon)x \tag{1.44}$$

We need to be cautious here since the first equality holds for $x \in \mathcal{D}(A)$, the set where A is defined, but the second holds for all $x \in X$. By the first equality, we have $x - S(\epsilon)x = \epsilon S(\epsilon)Ax \le \epsilon ||S(\epsilon)||\ ||Ax||$, thus

$$\lim_{\epsilon \to 0} S(\epsilon)x = x \tag{1.45}$$

Since $||S(\epsilon)|| \le C(1 - \epsilon\gamma)^{-m}||$, $\{S(\epsilon)\}$ is equicontinuous and thus the above equation holds for all $x \in X$. Now, let

$$T(t, \epsilon) = \exp\{tAS(\epsilon)\} \tag{1.46}$$

We claim that:

$$||T(t, \epsilon)|| \le C \exp\{\frac{\gamma t}{1 - \epsilon\gamma}\} \tag{1.47}$$

To show the claim. we first notice that from 1.44, we have $\epsilon AS(\epsilon) = S(\epsilon) - I$, which gives $tAS(\epsilon) = \frac{t}{\epsilon}(S(\epsilon) - I)$

$$\begin{aligned} ||T(t, \epsilon)|| &= ||e^{-t/\epsilon} \sum_{m=0}^\infty \frac{t^m S^m(\epsilon)}{\epsilon^m m!}|| \\ &\le e^{-t/\epsilon} \sum_{m=0}^\infty \frac{t^m C(1 - \epsilon\gamma)^{-m}}{\epsilon^m m!} \\ &= C \exp\{\frac{1}{1 - \epsilon\gamma}\} \le C \exp\{\frac{\gamma t}{1 - \epsilon\gamma}\} \end{aligned} \tag{1.48}$$

for $\gamma > 1/t,\ t > 0,\ 0 < \epsilon < \epsilon_0$. Now for $x \in \mathcal{D}(A)$, $\{T(t, \epsilon)T(t, \delta)^{-1}x\} = \exp\{tA(S(\epsilon) - S(\delta))\}x$. Thus,

$$\begin{aligned} \frac{d}{dt}\{T(t, \epsilon)T(t, \delta)^{-1}x\} &= A(S(\epsilon) - S(\delta) \exp\{tA(S(\epsilon) - S(\delta))\}x \\ &= T(t, \epsilon)T(t, \delta)^{-1}(S(\epsilon) - S(\delta)Ax \end{aligned} \tag{1.49}$$

Integrating on both sides from 0 to t gives:

$$T(t,\epsilon)T(t,\delta)^{-1}x - x = \int_0^t T(u,\epsilon)T(u,\delta)^{-1}(S(\epsilon) - S(\delta)Ax\ du \tag{1.50}$$

Applying $T(t,\delta)$ on both sides gives:

$$T(t,\epsilon)x - T(t,\delta)x = \int_0^t T(u,\epsilon)T(t-u,\delta)^{-1}(S(\epsilon) - S(\delta)Ax\ du \tag{1.51}$$

We note that the right hand side $\to 0$ as $\epsilon, \delta \to 0$. This shows that $T(t,\epsilon)$ is Cauchy as $\epsilon \to 0$. The completeness of Banach space implies that $\lim_{\epsilon\to 0} T(t,\epsilon)x$ exists for all $x \in \mathcal{D}(A)$ uniformly on every bounded subset of $[0,+\infty)$. We let $Q(t)x = \lim_{\epsilon\to 0} T(t,\epsilon)x$. 1.47 shows that $||Q(t)|| \le Ce^{\gamma t}$. By equicontinuity and $\mathcal{D}(A)$ dense in X, we have $Q(t) = \lim_{\epsilon\to 0} T(t,\epsilon)x$ defined for all $x \in X$. $Q(t)$ is a strongly continuous semigroup follows directly from the definition of $T(t,\epsilon)x$.

Finally, we check that A is indeed the infinitesimal generator of $\{Q(t)\}$. Let $\tilde{A}$ be the infinitesimal generator of $\{Q(t)\}$, then by part (6) of 1.10, we have:

$$(\lambda I - \tilde{A})^{-1}x = \int_0^\infty e^{-\gamma t}Q(t)x\ dt \tag{1.52}$$

Since we have $AS(\epsilon)$ the infinitesimal generator of $T(t,\epsilon)$, we have

$$(\lambda I - AS(\epsilon))^{-1}x = \int_0^\infty e^{-\gamma t}T(t,\epsilon)x\ dt \tag{1.53}$$

Taking the limit on both sides gives:

$$(\lambda I - A)^{-1}x = \int_0^\infty e^{-\gamma t}Q(t)x\ dt \tag{1.54}$$

Thus, we have:

$$(\lambda I - \tilde{A})^{-1}x = (\lambda I - A)^{-1}x \tag{1.55}$$

which shows that $\tilde{A} = A$, and finishes this proof. □

1.2. Application to Evolution Equations.

In the final section we present the application of the semigroup theory to the problem of well-posedness of evolutions equations. This should offer a glance at the role of semigroups in the theory of PDEs.

We consider the initial value problem:

$$\begin{cases} \dot{u}(t) = Au(t) & t \ge 0, \\ u(0) = x \end{cases} \tag{1.56}$$

where t represents time and $u(t)$ is a function with values in a Banach space X. $A : \mathcal{D}(A) \subset X \to X$ a linear operator and $x \in X$ the initial value. We call 1.56 the **abstract Cauchy problem (ACP)** associated to $(A, \mathcal{D}(A))$. A function $u : \overline{\mathbb{R}^+} \to X$ is called a **(classical) solution** of ACP if $u \in C^1(X)$, $u(t) \in \mathcal{D}(A)$ for all $t \ge 0$, and 1.56 holds.

If a strongly continuous semigroup $Q(t)$ is generated by $(A, \mathcal{D}(A))$, noticing that $\frac{d}{dt}Q(t)x = AQ(t)x$, we can check that $u(t) = Q(t)x$ is a solution to 1.56, and it actually is the unique solution, which is made rigorous by the following proposition. Thus, Hille-Yosida provides the precise condition when 1.56 has a solution.

Proposition 1.57. *Let $(A, \mathcal{D}(A))$ be the infinitesimal generator of a strongly continuous semigroup $\{Q(t)\}$, $t \geq 0$. Then, for every $x \in \mathcal{D}(A)$, the function*

$$u : t \mapsto u(t) := Q(t)x \tag{1.58}$$

is the unique solution to 1.56.

Proof. $Q(t)$ satisfies 1.56 by theorem 1.10 part *(4)*. We only need to prove the uniqueness. We first notice that if u is a solution to 1.56, then u satisfies the integral equation:

$$u(t) = A\int_0^t u(s)\ ds + x \tag{1.59}$$

It's sufficient to show that if $x = 0$, $u \equiv 0$. We consider:

$$\frac{d}{ds}Q(t-s)\int_0^s u(r)\ dr = -Q(t-s)A\int_0^s u(r)\ dr + Q(t-s)u(s)\ dt = 0 \tag{1.60}$$

Integrating both sides of above equation from 0 to t and using $Q(0) = I$ gives:

$$\int_0^s u(r)\ dr = 0 \quad \Rightarrow \quad u \equiv 0 \tag{1.61}$$

□

2. APPENDIX

The details filled by me and not presented in the reference:

The comment after the 1.3, Banach Steinhaus Theorem, the details filled in in the proof of the two theorems.

1.26 in Rudin is wrong (there shouldn't be e^t term), I corrected the calculation.

1.34 Rudin has a typo that is corrected ($e^{\epsilon\gamma-1} \to e^{\epsilon\gamma} - 1$)

The comment before and after 1.56 is written by me.

3. BIBLIOGRAPHY

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